Discrete Calculus

basic definitions and theorems

Abstract

This paper deals with real functions of an integer variable. Especially the change of the function-value depending on a change of its variable, and also the sum of function-values across a given integer variable-interval.

Notes

For all definitions and theorems that follow it's assumed two things.
(1) The stated functions are defined for all integers, or atleast for the used intervals.
(2) All function-variables used are integers.

Also, these definitions and theorems may or may not be the "standard" way of approaching the subject. They were however chosen for their simplification of certain problems, so eitherway they are atleast somewhat useful.

Definitions

D(discrete derivative)

$$f'(x) = f(x+1) - f(x)$$

Given a function, f(x), the discrete derivative of f in the point x, written f´(x), equals the change in f going from x to x+1.

D(discrete integral)

$$\sum_a^b f'(x)\Delta x = f(b) - f(a)$$

Theorems

T(derivative of sum)

$$\left(f(x) + g(x) \right)' = f'(x) + g'(x)$$

Proof

$$\left(f(x) + g(x) \right)' = \left(\text{D(discrete derivative)} \rightarrow \right) =$$
$$= \left(f(x+1) + g(x+1) \right) - \left(f(x) + g(x) \right) =$$
$$= \left(f(x+1) - f(x) \right) + \left(g(x+1) - g(x) \right) = f'(x) + g'(x)$$

T(derivative with constant factor)

$$\left(cf(x) \right)' = cf'(x)$$

Proof

$$\left(cf(x) \right)' = cf(x+1) - cf(x) = c\left(f(x+1) - f(x) \right) = cf'(x)$$

T(opposite integral)

$$\sum_a^b f'(x)\Delta x = -\sum_b^a f'(x)\Delta x$$

Proof

$$\sum_a^b f'(x)\Delta x = \left(D(\textit{discrete integral}) \rightarrow \right) = f(b) - f(a) =$$
$$= -\left(f(a) - f(b) \right) = -\sum_b^a f'(x)\Delta x$$

T(discrete jump)

$$\mathit{if} \quad x \geq 1 \quad \mathit{then} \quad f(a+x) = f(a) + \sum_{i=a}^{(a+x)-1} f'(i)$$

Proof

Suppose that for some choice of x>0, it is true that:

$$(1) \quad f(a+x) = f(a) + \sum_{i=a}^{(a+x)-1} f'(i)$$

then (1) is also true for x+1, beacuse:

$$f(a+x+1) = \left(D(d\,d) \to\right) = f(a+x) + f'(a+x) =$$

$$= \left((1) \to\right) = \left(f(a) + \sum_{i=a}^{(a+x)-1} f'(i) \right) + f'(a+x) =$$

$$= f(a) + \sum_{i=a}^{a+x} f'(i) = f(a) + \sum_{i=a}^{(a+x+1)-1} f'(i)$$

now, (1) is true for x=1, because:

$$f(a+1) = f(a) + f'(a) = f(a) + \sum_{i=a}^{(a+1)-1} f'(i)$$

therefore, by induction, (1) is true for all x>0.

T(equalities of the integral)

$$(1)\quad if \quad a<b \quad then \quad \sum_a^b f'(x)\Delta x = \sum_{i=a}^{b-1} f'(i)$$

$$(2)\quad if \quad a=b \quad then \quad \sum_a^b f'(x)\Delta x = 0$$

$$(3)\quad if \quad a>b \quad then \quad \sum_a^b f'(x)\Delta x = -\sum_{i=b}^{a-1} f'(i)$$

Proof of (1)

$$\sum_a^b f'(x)\Delta x = \big(D(d\,i) \rightarrow\big) = f(b) - f(a) =$$

$$= \big((a<b), T(d\ \ j)\, with\, (x = b-a) \rightarrow\big) =$$

$$= \left(f(a) + \sum_{i=a}^{(a+b-a)-1} f'(i) \right) - f(a) = \sum_{i=a}^{b-1} f'(i)$$

Proof of (2)

$$\sum_a^b f'(x)\Delta x = f(b) - f(a) = \big((a=b) \rightarrow\big) = f(a) - f(a) = 0$$

Proof of (3)

$$\sum_a^b f'(x)\Delta x = \big(T(o\,i) \rightarrow\big) = -\sum_b^a f'(x)\Delta x =$$

$$= \big((a>b), (1) \rightarrow\big) = -\sum_{i=b}^{a-1} f'(i)$$
